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DIGITAL FILTER DESIGNS THAT ARE FREE OF LIMIT CYCLES

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Finite word-length adders and quantizing methods used to construct digital filters can cause non-linearities to occur in the closed-loop filter system. Such non-linearities can degrade system performance in what is known as a limit cycle behavior of the output. Overflow oscillations are limit cycles that are induced by the repeated overflow of finite word-length adders used in filter realizations. It is possible to construct digital filters that are free of overflow oscillations. Non-linear analysis techniques can be used to derive conditions to govern digital filter designs that will be free of overflow oscillations. This paper will formulate a general theorem to ensure that nth order digital filters will be free of overflow oscillations. An implementation of digital filters with two's complement arithmetic adders is included to demonstrate the result of the analysis.

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DIGITAL FILTER DESIGNS THAT ARE FREE OF LIMIT CYCLES

Scott M. Bolen
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Abstract: Finite word-length adders and quantizing methods used to construct digital filters can cause non-linearities to occur in the closed-loop filter system. Such non-linearities can degrade system performance in what is known as a limit cycle behavior of the output. Overflow oscillations are limit cycles that are induced by the repeated overflow of finite word-length adders used in filter realizations. It is possible to construct digital filters that are free of overflow oscillations. Non-linear analysis techniques can be used to derive conditions to govern digital filter designs that will be free of overflow oscillations. This paper will formulate a general theorem to ensure that nth order digital filters will be free of overflow oscillations. An implementation of digital filters with two's complement arithmetic adders is included to demonstrate the result of the analysis.

I) INTRODUCTION

Digital filter systems are implemented using finite word-length arithmetic adders. Because these adders are physical devices of finite precision it is possible for overflow to occur in instances when large signals are added together. The overflow of finite precision adders is a non-linear operation. For repeated overflow the output of the filter can result in periodically recurring values that form a limit cycle.

Stable linear digital filters can be designed to be free of overflow oscillations. Conditions can be imposed on the state variable equation of the linear filter to ensure that overflow non-linearities cannot exist. There are two basic approaches to the analysis which derive such conditions. The most popular approach has been the Lyapunov method. The other approach is to use a non-linear frequency domain criteria similar to the circle or Popov criteria.

In 1969, Eber, Mazo, and Taylor [3] presented a comprehensive investigation of limit cycles induced by adder overflow. They characterized conditions for which second order digital filters can remain free of limit cycles. In 1978, Mills, Mullis, and Roberts [1] were able to extend this earlier work to include two theorems that guarantee digital filter realizations that are free of overflow oscillations. Also, in 1975, Claasen, Mecklenbrauker, and Peek [5] used the non-linear frequency analysis criteria to develop filters that do not exhibit limit cycles.

II) LYAPUNOV APPROACH [1]

A stable linear digital filter can be represented by the equation:

$$X(t+1) = AX(t) + bu(t). \qquad (1)$$

When the digital filter is implemented the linear filter system is imposed with non-linearities and becomes the non-linear system given by:

$$X(t+1) = F[X(t),u(t)]. \qquad (2)$$

The state space of the digital filter can be described as:

$$C = \{X \in R^n: |Xi| < 1 \text{ for each } i\}. \tag{3}$$

When the trajectories of the linear filter system, eq (1), remain inside the state space described by C, then a zero-input digital filter behaves properly, that is, no overflow oscillations will occur at the output of the filter. However, when the trajectories of the filter system leave the state space of the filter overflow oscillations can occur that will induce a limit cycle behavior.

Consider the non-linear digital filter model described by Mills, Mullis, and Roberts:

$$F(X,u) = H(AX + bu) \tag{4}$$

where

$$|H(X)|i = h(Xi) \tag{5}$$

and the non-linearity h(*) must satisfy the conditions:

$$\text{(i)} \quad |h(v)| < 1 \text{ for all } v$$

$$\text{(ii)} \quad h(v) = v \text{ if } |v| < 1. \tag{6}$$

These two conditions imply that

$$|h(v)| < |v| \text{ for all } v. \tag{7}$$

This model is valid when the non-linearity h(*) is a two's complement characteristic and the filter is given in its state variable form. Conditions for digital filters free of overflow oscillations will be developed from this model. For now the effect of quantization errors will be neglected.

For u(t) = 0 the digital filter model reduces to a zero-input system. The zero-input non-linear filter system can be written as:

$$X(t+1) = H[AX(t)]. \tag{8}$$

Mills, Mullis, and Roberts have derived results that test for the existence of overflow oscillations and provide conditions that guarantee such periodic solutions cannot occur in digital filter realizations. Their results are stated in two theorems (see also Appendix):

Theorem 1: If there exists a diagonal matrix D with positive elements such that the matrix equation D - A'DA is positive definite then overflow oscillations will be impossible. The only periodic solution of the system

$$X(t+1) = H[AX(t)]$$

is the identically zero solution.

Theorem 2: Theorem 2 gives conditions for which second-order digital filters will be free of overflow oscillations. For a 2x2 matrix A whose eigenvalues satisfy $|\lambda| < 1$, there exists a positive definite diagonal matrix D
iff

$$\text{(i)} \quad (a12)(a21) \geq 0 \tag{9}$$

or if (a12)(a21) < 0, then

$$\text{(ii)} \quad |a11 - a22| + \det(A) < 1. \tag{10}$$

The conditions of theorem 2 can be extended to form a more general result. For an nth order digital filter consider theorem 3.

Theorem 3: For an nxn matrix A whose eigenvalues satisfy $|\lambda| < 1$, there exists a positive definite diagonal matrix D iff

(i) Permutations of (aii)(ajj) > 0 (11)
for i,j = 1,2,3,...,n

(ii) $(\sum_{i=1}^{n} aii)^2 + 2 \sum$ (Permutations of (aii)(ajj)) $- [\det(A)]^2 < 1$. (12)

For any stable linear digital filter there exists a two's complement implementation that will be free of overflow oscillations. From the Lyapunov stability theorem it is known that the unique solution P to the matrix equation

$$P^{-1} = A'PA + I$$

is positive definite. If T is a symmetric square root of P^{-1} then,

$$P^{-1} = I - (T^{-1}AT)'I(T^{-1}AT)$$

Hence, the coordinate transformation T will produce a new A that will meet the conditions of theorem 1.

Example 1:

From theorem 2, digital filters can be designed that are free of overflow oscillations. For example, consider the second-order digital filter of figure 1 [2]:

$$A = \begin{bmatrix} 0 & 1 \\ -b2 & -b1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} m(n-2) \\ m(n-1) \end{bmatrix}$$

From the conditions of theorem 2, the filter is free of overflow oscillations if:

$$(a12)(a21) = -b2 \geq 0$$

or if

$$(a12)(a21) = -b2 < 0,$$

then

$$|a11 - a22| + \det(A) = b1 + b2 < 1.$$

Therefore, overflow oscillations are absent if

$$|b1| + |b2| < 1.$$

This example shows how theorem 2 can be used to design "direct form" digital filters (see also figure 2).

Example 2:

Digital filters with the property A'A = AA' are called "normal". Filters of normal form are free of overflow oscillations; this is evident form theorem 1. Choose D = I, then the conditions for which overflow oscillations cannot occur will be satisfied. This is true from the spectral property of normal matrices. The eigenvalues of I - A'A are $\{1 - \lambda_i^2\}$. Since A is stable, the eigenvalues of I - A'A are positive. Hence, the symmetric matrix I - A'A is positive definite. Other important features of normal filters is their low sensitivity to parameter quantization, low round-off noise, and they are invariant under frequency transformation.

III. FREQUENCY ANALYSIS APPROACH [5]

Again, consider the stable linear filter system described by eq(1). In the realization of this filter quantizers are used to keep signals form exceeding the finite precision of the practical devices used in the system. Quantization is the truncation or rounding of a signal to an arbitrary precision. Quantization of signals is a non-linear operation which induces limit cycles to occur in the closed-loop filter system. The model for digital filters with quantization errors is shown in figure 3. The non-linear system has a linear part W and a non-linearity Q. Claasen, Mecklenbruaker, and Peek have used frequency analysis techniques similar to the Popov and circle criteria to develop conditions for which zero-input digital filters will be free of limit cycles. Their results extend the earlier work done by A.I. Barkin in 1970 [6].

If the non-linearity Q is a sector bounded non-linearity specified by (see also figure 4):

$$Q(0) = 0 \tag{13a}$$

and

$$0 < \frac{Q(x)}{x} < k, \quad \text{for all } x \neq 0 \tag{13b}$$

then conditions can be derived on the state variable equation of a digital filter to ensure that limit cycles will not occur in the zero-input, closed-loop filter system. A number of theorems are presented by Claasen, Mecklenbruaker, and Peek to summarize these conditions. The appendix contains proofs of the theorems.

Theorem 4: For a digital filter system with a non-linearity given by equation (1), with linear part W(Z) which is finite for $|Z| = 1$, and sector non-linearity that satifies equations (13a) and (13b), then limit cycles of length L are absent if:

$$\mathrm{Re}\{W(Zl)\} - 1/k < 0 \qquad \text{for } l = 0,1,\ldots,L/2 \tag{14}$$

were

$$Zl = \exp\{j(2\pi/L)l\}.$$

To apply theorem 4 consider the following example:

Example 3:

For implementation of the filter system, choose the quantizer such that for $k = 1$ truncation occurs and for $k = 2$ rounding occurs, then for the digital filter shown in figure 5,

$$W(Z) = (b1)z^{-1} + (b2)z^{-2}$$

and

$$Re\{W(Z)\} = (b1)\cos[(2\pi/L)1] + (b2)\cos[(2\pi/L)1].$$

Limit cycles of length $L = 1$ cannot exist if

$$b1 + b2 - 1/k < 0.$$

Moreover, all limit cycles are absent from the digital filter system if

$$(b1)\cos(\phi) + (b2)\cos(\phi) - 1/k < 0, \quad 0 < \phi < 2 \qquad (15)$$

These results are shown in figure 6a. Filter coefficients chosen in the shaded area will guarantee a filter that is free of limit cycles.

The remaining theorems will extend the concept put forth by theorem 4. By imposing extra constraints on the non-linearity of the system a greater number of coefficients can be selected to provide filters that are absent of limit cycles.

Theorem 5: For a system with a non-linearity that satisfies the conditions of equation (13) and the additional constraint:

$$\{Q(x+h) - Q(x)\}\ h \geq 0, \quad \text{for all } x,h \qquad (16)$$

if there exists an $a > 0$ such that $1 = 0,1,\ldots,L/2$ and

$$Re\{W(Z1)[1 + \quad a(1 - Z1^{P})]\} - 1/k < 0 \qquad (17)$$

where $Z = \exp[j(2\pi/L)1]$, then limit cycles of length L will be absent. Figure 6b shows all possible filter coefficients that satisfy the conditions of the theorem.

Theorem 6: For a system with a symmetric non-linearity characteristic that also satisfies the conditions of equations (13) and (16) if there exists some arbitrary constants $a,b > 0$ such that for $1 = 0,1,\ldots,L/2$

$$Re\{W(Z1)[1 + \quad (a(1 - Z1^{P}) + \ b(1 + Z1^{P}))]\} - 1/k < 0 \qquad (18)$$

were $Z1 = \exp\{j(2\pi/L)1\}$, then limit cycles of length L cannot exist. Figure 6c shows the possible filter coefficients that will satisfy the conditions of the theorem.

the symmetric non-linear characteristic of theorem 6 implies that

$$Q(-x) = -Q(x).$$

To apply the results of theorems 5 and 6 it is necessary to use linear programming techniques to find solutions for the filter coefficients. In each of the new theorems it is important to note that with additional constraints imposed on the non-linearity of the system a larger number of filter realizations exist that cannot have limit cycles. The next set of theorems will generalize the results of this section.

In practice digital filter systems can contain several non-linearities. It is possible to model such systems by extending the results of the previous theorems. The sector boundary condition of each non-linearity is arbitrary such that the following conditions are valid:

$$Qn(0) = 0 \qquad (19)$$

and

$$0 < \frac{Q_n(x)}{x} < kn, \quad \text{for every } x = 0. \qquad (20)$$

The next set of theorems will be developed from these contraints.

Theorem 7: For a system with n non-linearities that satisfy the conditions of equations (19) and (20) if for $l = 0,1,\ldots,L/2$ the Hermitian part of

$$W(Zl) - \mathrm{diag}(1/k) \qquad (21)$$

is negative definite, then limit cycles of length L cannot exist.

Theorem 8: For a system with *n* non-linearities satisifing the conditions of equations (19) & (20) and the additional constraint that:

$$\{Q(x+h) - Q(x)\}h \geq 0, \quad \text{for every } x,h \qquad (22)$$

if for $l = 0,1,\ldots,L/2$ and the Hermitian part of

$$\{1 + \mathrm{diag}[(a(1 - Zl^P) + b(1 + Zl^P)]\}W(Zl) - \mathrm{diag}(1/k) \qquad (23)$$

is negative definite, then limit cycles of length L cannot exist.

For nth order digital filters with n non-linearities theorem 4 can be used to determine the filter coefficients that would allow the system to be free of limit cycles. *Consider the second-order filter shown in* figure 7. Since the linear part is:

$$W(Zl) = \begin{bmatrix} az^{-1} & az^{-1} \\ bz^{-2} & bz^{-2} \end{bmatrix}$$

the Hermitian part of $W(Zl) - (1/k)I$ is given by:

$$(1/2)\begin{bmatrix} az^{-1} + az - 2/k & az^{-1} + bz^{2} \\ az + bz^{-2} & bz^{-2} + bz^{2} - 2/k \end{bmatrix} \qquad (24)$$

For the filter to be free of limit cycles of any length the Hermitian matrix given in eq(24) becomes

$$(1/2)\begin{bmatrix} 2a\cos(\ ) - 2/k & ae^{-j} + be^{j2} \\ 2ae^{j} + be^{-j2} & 2b\cos(2\ ) - 2/k \end{bmatrix}$$

which is negative definite for $0 < \ < 2$. Figure 8 shows the possible coefficients that satisfy this condition.

IV. SUMMARY

When finite word-length adders are used to implement digital filter designs non-linearities can occur in the closed-loop filter system. These non-linearities are caused by the overflow that can occur during the addition operation. Limit cycles induced by the repeated overflow of finite word-length adders are called overflow oscillations.

It is possible to construct digital filters that are free of overflow oscillations. Mills, Mullis, and Roberts established a set of theorems to determine the existence of such limit cycles and provided

conditions to eliminate limit cycles from the filter realizations. Also, they reported that a class of digital filters exists that will not supprot limit cycle behavior. Filters with the special property $AA' = A'A$ are free of limit cycles. Filters of this design are called normal digital filters.

The non-linear operation associated with quantization can cause limit cycles to persist in the closed-loop filter system. Classen, Mecklenbrauker, and Peek used frequency analysis techniques similar to the Popov and circle criteria to derive conditions for which digital filters will be free of limit cycles. Their results were generalized by the author to provide conditions for nth order filters that will be free of the limit cycle behavior.

APPENDIX:

this appendix contains the proofs of the theorems presented in this paper.

A) **Theorem 1:**

proof:

Consider a zero-input digital filter; $u(t) = 0$. The non-linear filter system is a composition of two parts; a linear part $X(t) = AX(t)$ and the application of the overflow non-linearity $h(*)$ on the vector $AX(t)$. The existence test involves finding a norm under which each operation is norm decreasing. That is,

$$||AX|| \leq r||X|| \quad \text{for all } X \text{ with } r < 1 \tag{A1}$$

$$||H(x)|| \leq ||X|| \quad \text{for all } X \tag{A2}$$

If these conditions are true, then it follows that the zero-input trajectories of the non-linear filter system

$$X(t+1) = H[X(t), u(t)]$$

tend to zero exponentially fast, that is

$$||X(t)|| \leq r^t||X(0)||.$$

Hence, the only periodic solution to

$$X(t+1) = H[AX(t)]$$

is the identically zero solution.

For a quadratic norm defined by

$$||X|| = [X'DX]^{1/2},$$

where X' denotes the transpose of the matrix X, in order for equation (A2) to be true, for all overflow characteristics $h(*)$, the matrix D must be necessarily diagonal. This condition is also sufficient since,

$$\begin{aligned} ||H(X)|| &= \sum_{i=1}^{n} (dii)\ g(xi)^2 \\ &= \sum_{i=1}^{n} (dii)\ xi^2 \\ &= \quad ||X||^2. \end{aligned}$$

Then, for equation (A2) to be true, it is necessary and sufficient that the matrix

$$Q = D - A'DA$$

be positive definite, since

$$r^2||x||^2 - ||Ax|| = X'[Q - (1-r^2)D]X.$$

If Q is positive definite then so is $Q - (1-r^2)D$ for r sufficiently close to unity. Therefore, the contitions of the theorem are valid iff there is a diabonal matrix D with positive diagonal elements such that the matrix D - A'DA is positive definite.

B) Theorem 2:

proof:

If there exists a positive diagonal matrix D such that D - A'DA is positive definite then there also exists a non-singular diagonal matrix T for which I- M is positive definite, where

$$M = (T^{-1}AT)'(T^{-1}AT).$$

The matrix (I - M) is positive definite iff the Tr(I - M) is positive and the det(I - M) is positive (ie the eigenvalues of (I - M) are positive). Consider,

$$\begin{aligned}
\det(I - M) &= \det(zI - M) \quad \text{for } z = 1 \\
&= 1 - Tr(M) + \det(M) \\
&= 1 - Tr(M) + [\det(A)]^2 \\
Tr(I - M) &= 2 - Tr(M) \\
&> 1 + [\det(A)]^2 - Tr(M) = \det(I - M)
\end{aligned}$$

So, consider only the det(I - M).

$$\det(I - M) = \det(A) - [a11^2 + a12^2 + a21^2/t + a22^2] \qquad \text{(A3)}$$

where t = T22/T11. Maximizing the right hand side of (A3) with respect to t yields the inequality:

$$\det(I - M) < 1 + [\det(A)]^2 - [a11^2 + 2(a12)(a21) + a22^2] \qquad \text{(A4)}$$

$$= [1 + \det(A)]^2 - [Tr(A)]^2 + 2(a12)(a21) - |(a12)(a21)|) \qquad \text{(A5)}$$

If the right hand side of (A5) is positive then the ocnditions of theorem 2 are valid. If (a12 a21) > 0 then the right hand side is the product of [1 + Tr(A) + det(A)][1 - Tr(A) + det(A)]. Ebert, Mazo, and Taylor have shown this quantity to be positive [3]. If (a12 a21) < 0, then the right hand side of equation (A5) is:

$$1 + [\det(A)]^2 - [a11^2 - 2(a12)(a21) + a22^2]$$

$$= [1 - \det(A)]^2 - (a11 - a22)^2.$$

This quantity is positive definite iff $|a11 - a22| + \det(A) < 1$. Therefore, the conditions of theorem 2 are valid.

C) Theorem 3:

proof:

The proof of theorem 3 follows form the proof of theorem 2. If there exists a positive definite diagonal matrix D such that D - A'DA is positive definite then there also exists a non-singular diagonal matrix T for which the matrix (I - M) is positive definite, where

$$M = (T^{-1}AT)'(T^{-1}AT)$$

The matrix (I - M) is positive definite iff Tr(I - M) is positive and det(I - M) is positive. Consider,

$$\det(I - M) = 1 - Tr(M) + [\det(A)]^2$$
$$= Tr(I - M).$$

In terms of the elements of T,

$$\det(I - M) = 1 + [\det(A)]^2 - (\quad aii)^2 - (T22/T11)a12^2 - (T33/T11)a13^2 - \ldots - (\text{Permutations of } (Tii/Tjj)(aii)(ajj))$$

Maximize the right hand side with respect to each Tii/Tjj then,

$$\det(I - M) = [1 + \det(A)]^2 - [Tr(A)]^2 + (\sum \text{Permutations of } [(aii)(ajj) - |(aii)(ajj)|]).$$

If each (aii ajj) > 0, then from the proof of theorem 2; [1 + det(A)] - [Tr(A)] is a positive quantity. if each (aii ajj) < 0, then det(I - M) is positive if

$$1 > (\sum_{i=1}^{n} aii)^2 + 2\{\sum [\text{Permutations of } (aii)(ajj)]\} - [\det(A)]^2.$$

D) Theorem 4:

proof:

Claasen, Mecklenbrauker, and Peek used the proof from A.I. Barkin to develop the proof for theorem 4. From Barkin, the following summation was used:

$$p = (2/L)\{\sum_{n=0}^{L-1} Q(Xn)[Xn - (1/X)(Q(Xn))]\}. \qquad (A6)$$

This function describe a zero-input digital filter system. Since the system non-linearity satisfies the ocnditions of equation (13), then

$$p > 0. \qquad (A7)$$

The proof of this theorem is by contradiction. Suppose that X is periodic with period length L. From Parseval's theorem equation (A6) can be transformed into a Fourier representation:

$$p = \sum_{l=0}^{L-1} (Y^*l)[Xl - (1/k)(Yl)]. \qquad \text{(A8)}$$

Also, the non-linear system is given as:

$$Xl = W(Zl)\ Yl. \qquad \text{(A9)}$$

So, from equations (A7) and (A8),

$$p = \sum_{l=0}^{L-1} |Yl|^2 [W(Zl) - 1/k].$$

This sum is always negative, which contradicts equation (A7), hence limit cycles of length L cannot exist.

E) Theorem 5:

proof:

The proof of this theorem is an extension of the proof of theorem 4. Consider the summation:

$$p(a1,\ldots,an) = (2/N)\{\sum_{n=0}^{L-1} Q(Xn)\ [Xn - (1/k)\ Q(Xn)] + \sum_{p=0}^{L-1} a \sum_{n=0}^{L-1} Q(Xn)\ [Xn - Xn{+}p]\}. \qquad \text{(A10)}$$

The proof of this theorem is by contradiction. Suppose that X is periodic with period of length L. The first term of the function is always positive. This is a result taken from Barkin [6]. The second term can be shown to be of the same form as the first term,

$$\begin{aligned} psp &= \sum_{n=0}^{p-1} Q(Xn)\ (Xn - Xn{+}p) \\ &= \sum_{i=0}^{p-1}\sum_{k=0}^{L-1} Q(Xkp{+}1)\ (Xkp{+}1 - Xkp{+}p{+}i) \\ &= \sum_{k=0}^{L-1} Q(Zk(p,i))\ [Zk(p,i) - Zk{+}1(p,i)], \end{aligned}$$

hence from Barkin this term is also positive. Therefore, it has been shown that $p(a1,\ldots,an)$ is positive. Apply Parseval's theorem to the function of eq (A10). The result is:

$$p = \sum_{l=0}^{L-1} |Yl|^2\ [Re(W(Zl))\ \{1 + \sum_{p=1}^{L-1} k(1 - Z^P)\} - (1/k)] \qquad \text{(A11)}$$

If it is possible for limit cycles to occur then the right hand side of equaiton (A10) is always negative, which is a contradiction. Therefore, limit cycles ov length L cannot exist under the conditions of specified by this theorem.

F) Theorem 6:

proof:

For the proof of this theorem add the term:

$$\sum_{p=1}^{L-1} b\ [\sum_{n=1}^{L-1} Q(Xn)\ (Xn + Xn{+}p)]$$

to the function specified in equaiton (A10). Now it is necessary to show that htis new term is non-negative for the abritrary constant b >

0. The inequality given by equation (18) is derived by applying Parsavel's theorem to the new function $p(a1,...,an)$.

The term

$$Sp = \sum_{n=0}^{L-1} Q(Xn)\ (Xn + Xn+p)$$

can be shown to be the same as:

$$\sum_{i=0}^{p-1} \sum_{k=0}^{Z_L-1} [Q(Zk(p,i))\ (Zk(p,i) - Zk+1(p,i)]. \qquad \text{(A12)}$$

Again, form Barkin this term is always non-negative, hence the new funciton will also always be non-negative. Therefore, for the conditions specified in theorem 6, limit cycles of length L cannot exist.

G) Theorem 7:

proof:

Consider the function of equation (A6), except that n non-linearities are introduced, equation (A6) becomes:

$$pn = (2/L)\ \{\sum_{l=0}^{L-1} Qn(Xm^n)\ (Xm^n - (1/k)(Qm(Xm^n))\} \qquad \text{(A13)}$$

From the proof of theorem 4 it follows that

$$pn \geq 0.$$

This means that the sum over the number of non-linearities of pn is non-negative. The proof of this theorem *is* by contradiction. Suppose that X is periodic. Recall that for the non-linear system:

$$X = W(Z1)\ Y. \qquad \text{(A14)}$$

So, applying Parseval's theorem to the function and from equation (A14) a similar result to theorem 4 is obtained:

$$p = \sum_{l=0}^{L-1} Yi\ [-\ diag(1/kn + W(Z1)]$$

for

$$A(Z) = -\ diag(1/km) + W(Z)$$

it follows that p is negative definite since the Hermitian part of A(Z) is negative for all 1. This is a contradiction to the previous statements. Therefore, limit cycles cannot exist under the conditions of theorem 7.

H) Theorem 8:

proof:

The proof of theorem 8 follows from the proofs of theorems 3 and 4. Adding the term:

$$(W(Z) - diag(1/kn))\ [diag(\sum_{p=1}^{L-1} \{a(1 - Z^p) + b(1 - Z^p)\})]$$

to equation (21), it is necessary to show that:

$$\sum_{p=1}^{L-1} \{a(1 - Z^p) + b(1 - Z^p)$$

is positive definite for the arbitrary constants a, $b > 0$, and the symmetric non-linearity of the system. Since if the above function is positive definite for the given conditions then from theorem 4 limit cycles cannot exist under the constraints of theorem 5.

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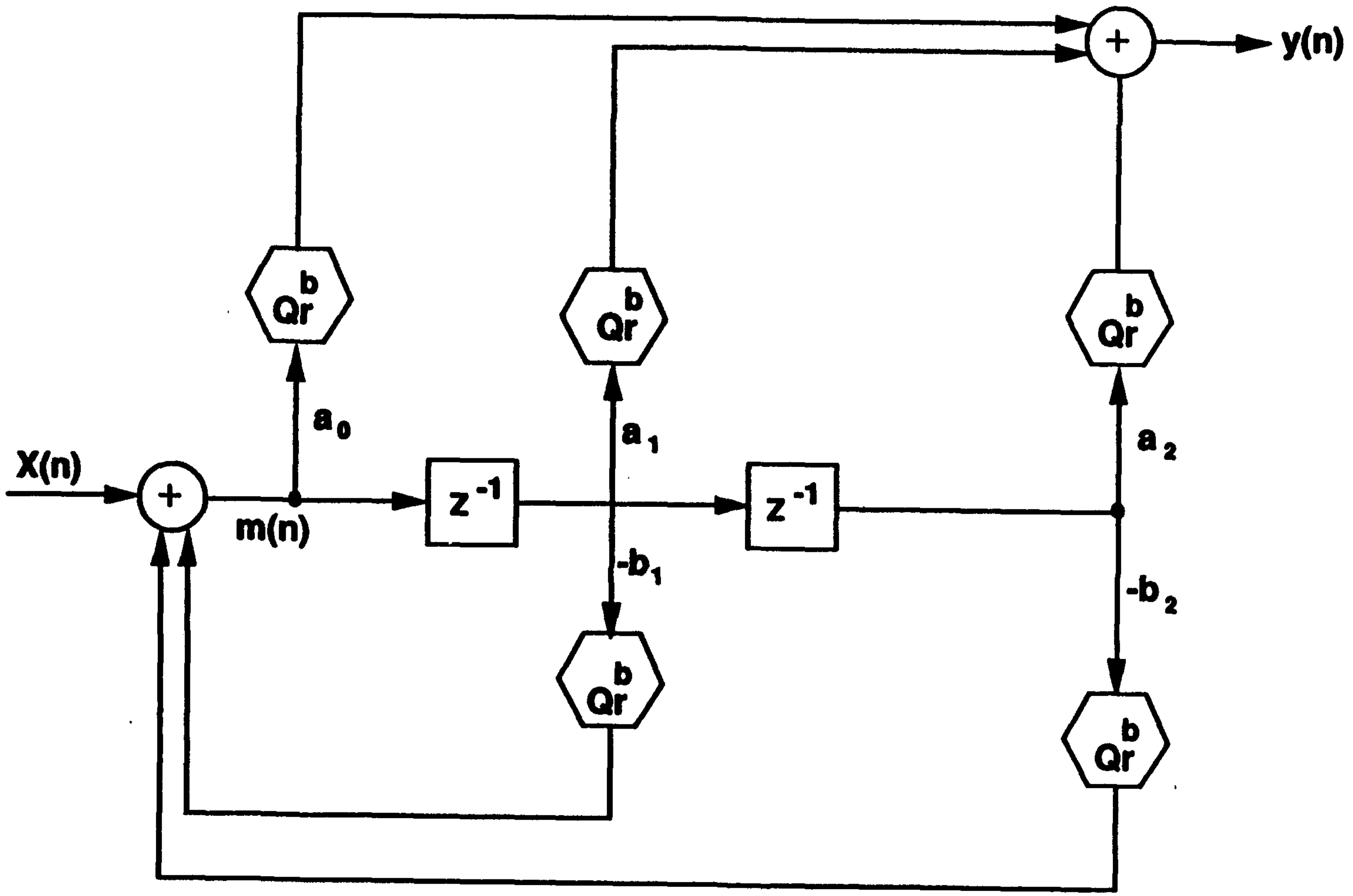

Figure 1. Second - order 1D filter with one quantizer per multiplier.

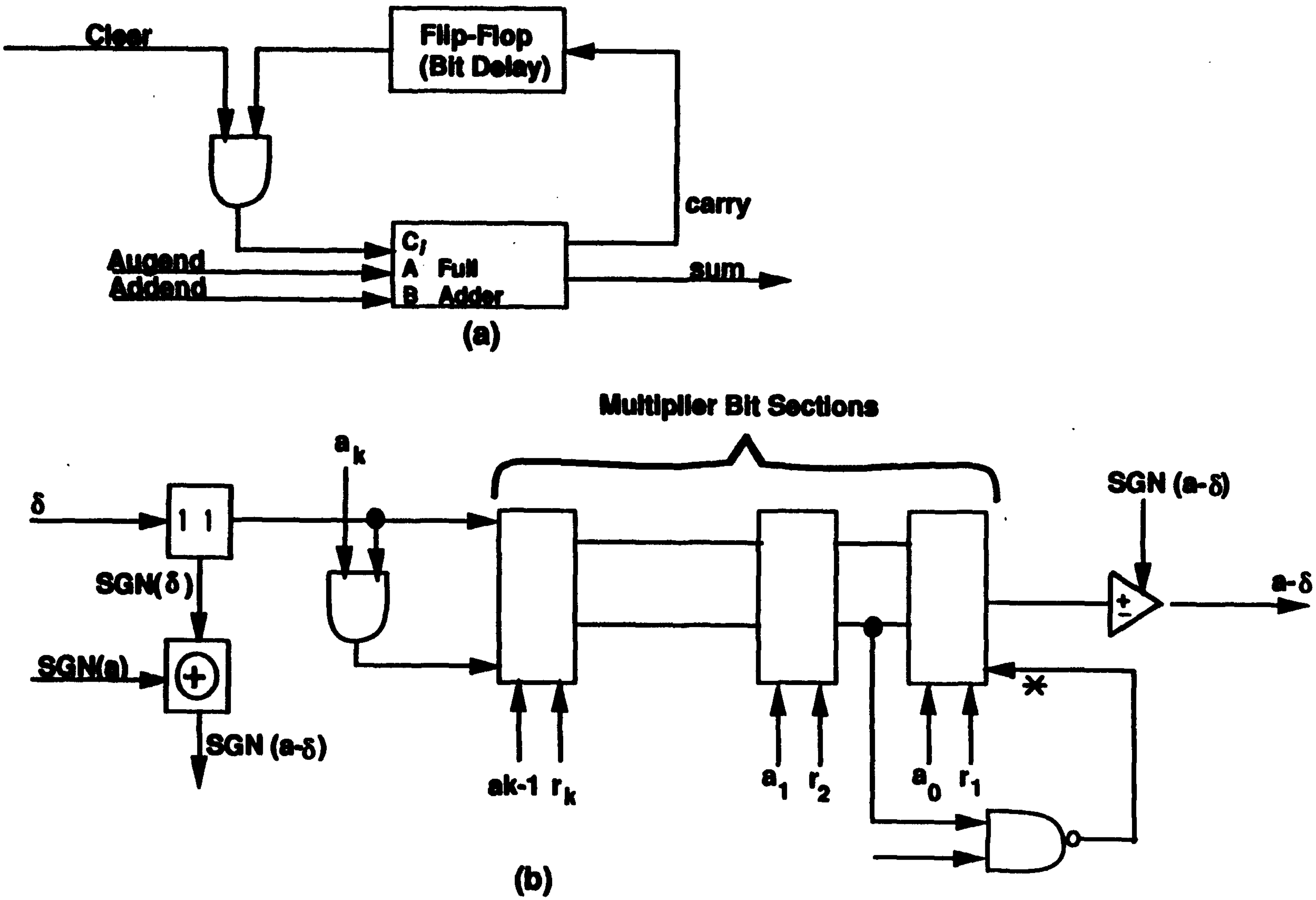

Figure 2a. Components of the digital filter realization:
a) Serial two's complement adder;
b) Serial Multiplier [4]

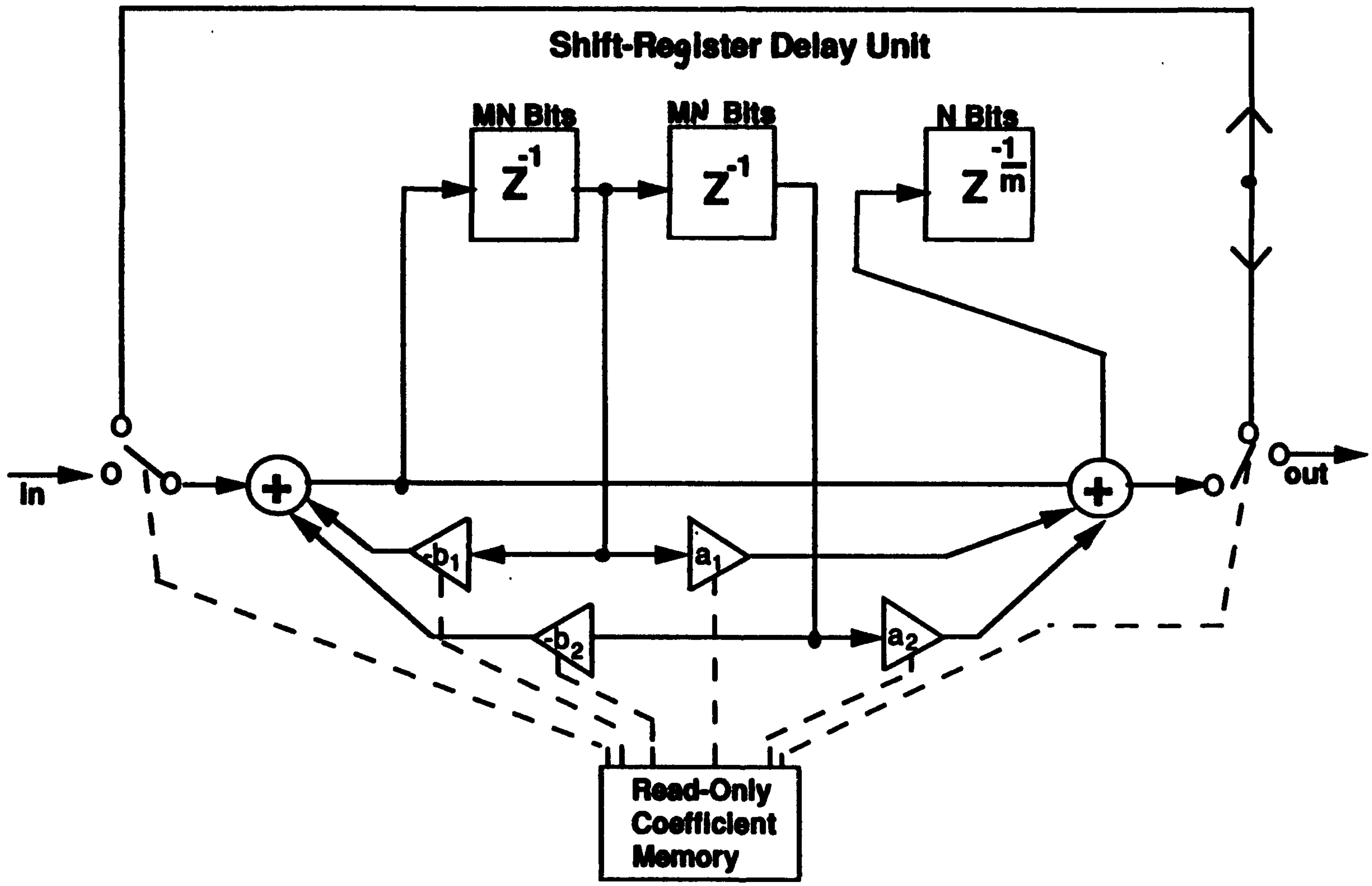

Figure 2b. General Second-order digital filter (hardware implementation) [4].

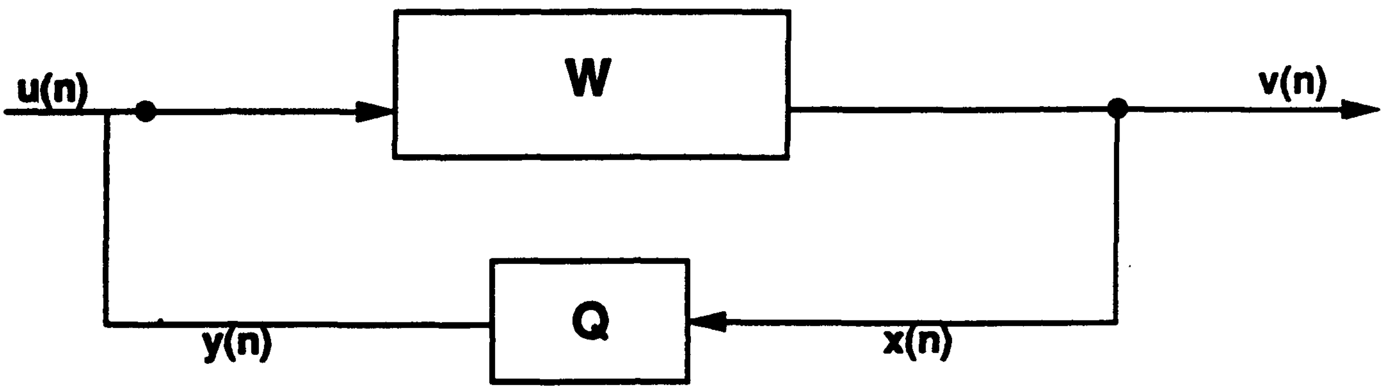

Figure 3. Digital filter system with a linear part W and nonlinearity Q.

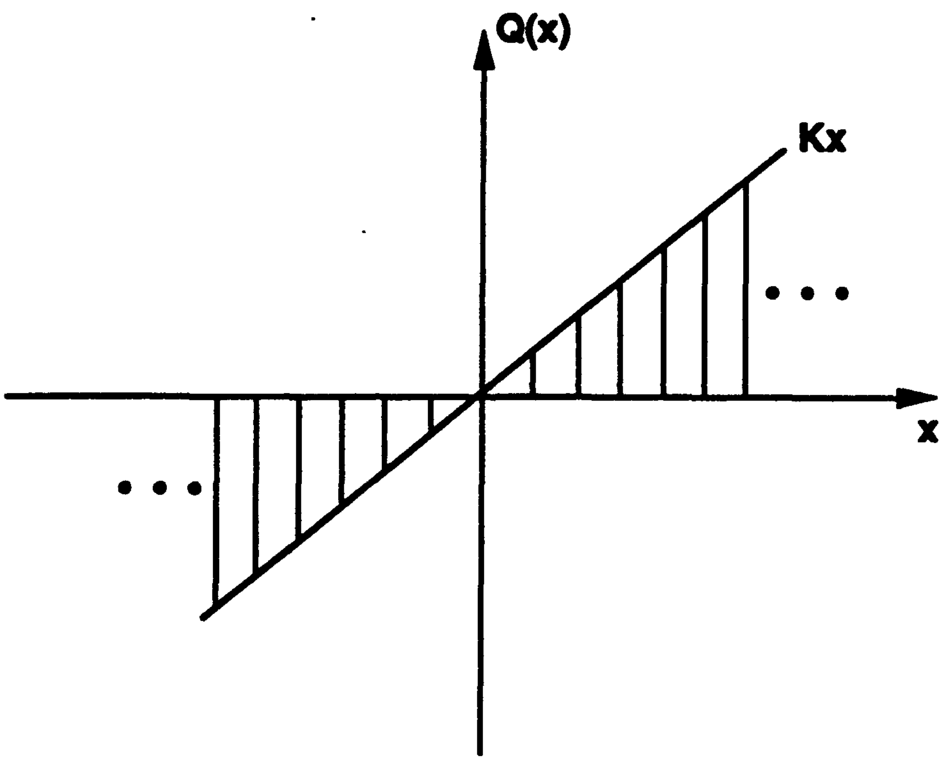

Figure 4. The nonlinearity Q(-) is bounded in the sector [o,k].

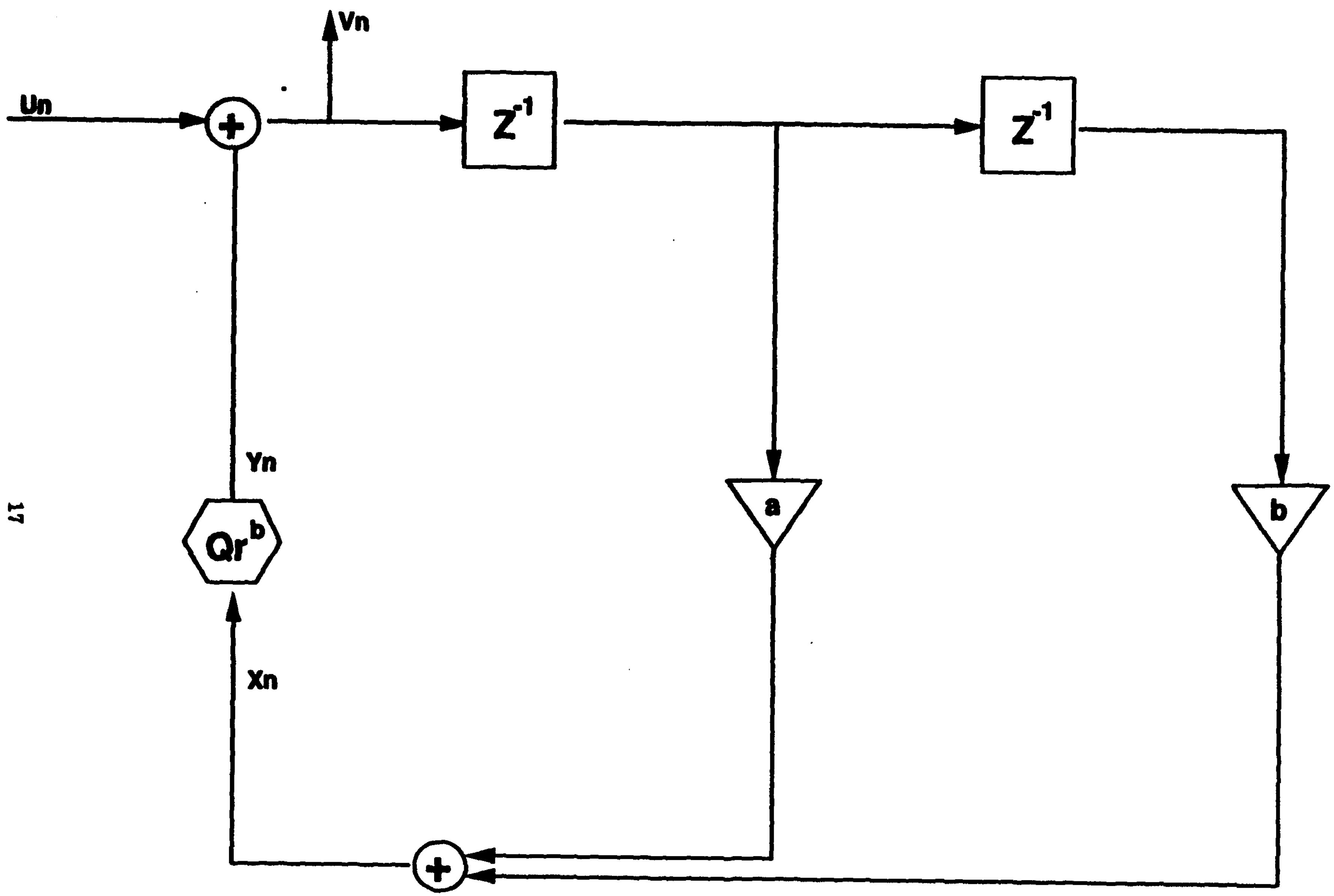

Figure 5. Second-order digital filter with one nonlinearity.

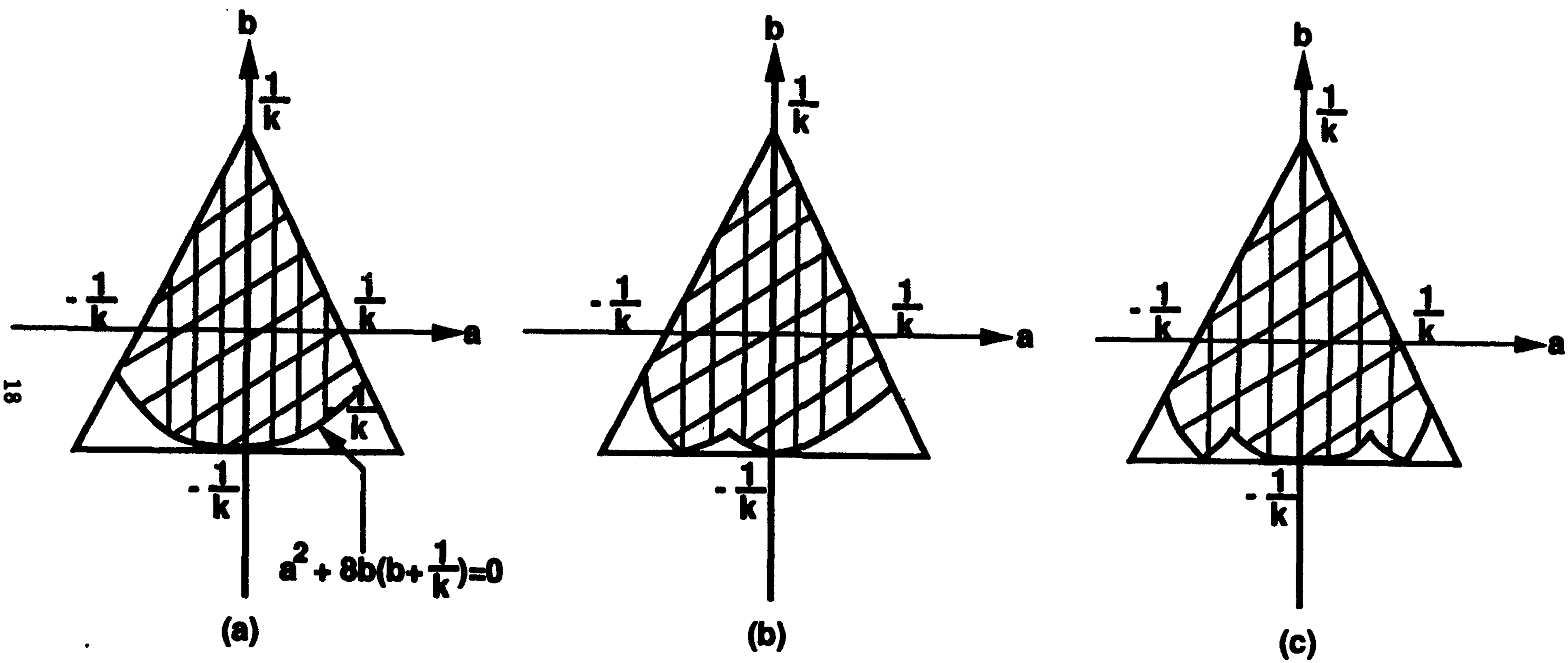

Figure 6. Area of asympototic stability of the digital filter shown in figure 5: (a) derived by theorem 4; (b) derived by theorem 5; (c) derived by theorem 6.

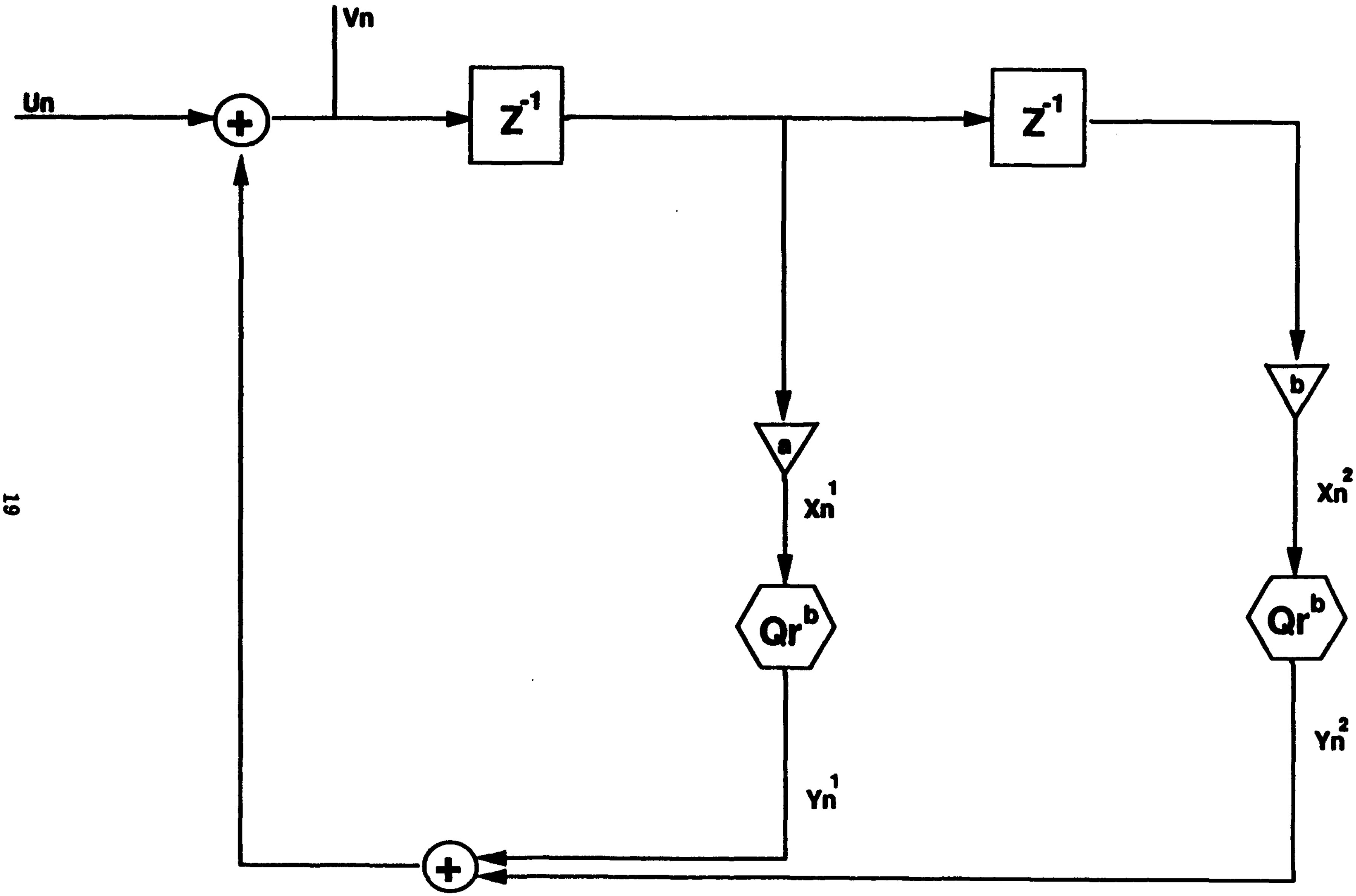

Figure 7. Second-order digital filter with two nonlinearities.

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